# Anisotropic cosmologies containing isotropic background radiation

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We present an anisotropic cosmological model based on a new exact solution of Einstein equations. The matter content consists of an anisotropic scalar field minimally coupled to gravity and of two isotropic perfect fluids that represent dust matter and radiation. The spacetime is described by a spatially homogeneous, Bianchi type III metric with a conformal expansion. The model respects the evolution of the scale factor predicted by standard cosmology, as well as the isotropy of the cosmic microwave background. Remarkably, the introduction of the scalar field, apart from explaining the spacetime anisotropy, gives rise to an energy density that is close to the critical density. As a consequence, the model is quasiflat during the entire history of the universe. Using these results, we are also able to construct approximate solutions for shear-free cosmological models with rotation. We finally carry out a quantitative discussion of the validity of such solutions, showing that our approximations are acceptably good if the angular velocity of the universe is within the observational bounds derived from rotation of galaxies.

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## I. INTRODUCTION

One of the best established facts in observational cosmology is the isotropy of the cosmic microwave background (CMB) [1]. This high degree of isotropy explains the success of cosmological perturbation theory [2] in reproducing the spectrum of anisotropies detected in the CMB [3]. The measurement of these anisotropies, originated from primordial fluctuations, has played a fundamental role in the advent of precision cosmology, allowing for the first time the determination of several cosmological parameters and the rejection of a large number of cosmological models [4].

The isotropy of the CMB, together with the apparent homogeneity and isotropy of clustering matter, smeared out over scales of the order of 100 Mpc [5], provide the main experimental support for the cosmological principle [6]. The spatial homogeneity and isotropy of the universe is incorporated in the standard cosmological model by using the Friedmann-Robertson-Walker (FRW) family of metrics to describe the spacetime. It should be clear, nevertheless, that the cosmological principle is actually a reasonable and fruitful hypothesis, rather than a proven fact. In order to clarify this issue, at least from a conceptual point of view, we want to show that it is possible to construct spatially homogeneous cosmologies which are anisotropic but still compatible with the observed isotropy of the background radiation and the matter with strong clustering properties.

Our starting point will be a spatially homogeneous spacetime metric of the Bianchi type III [7] subject to conformal expansion. As shown in Ref. [8], the condition that the expansion be conformal is crucial for the isotropy of the CMB. Part of the matter content will be given by a two-component perfect fluid describing the (idealized) radiation and dust matter that are present in our universe. By its own, this perfect fluid cannot account for the anisotropy of the spacetime [9]. However, we will show that this problem can be solved just by including an additional matter source consisting in an anisotropic scalar field. In this way, we will be able to construct an exact solution of Einstein equations with the required properties.

In addition to their important role in inflationary models [10] and Brans-Dicke cosmologies [6,11], the use of scalar fields in cosmology has received renewed attention during recent years. The observed relation between luminosity distance and redshift for type Ia supernovae (SNe Ia) has supplied strong evidence in favor of an accelerated expansion of the universe [12]. In order to explain this acceleration and fit the SNe Ia data, cosmological models with a new matter component have been proposed. This component, called quintessence, can be modeled by a light scalar field with a self-interaction potential and with minimal coupling to gravity [13]. Comparison of the theoretical predictions with SNe Ia and CMB observations leads to an estimate for the dark energy density of this quintessence field that (roughly speaking) is of the order of magnitude of the critical density [12,14].

From this perspective, our approach to construct anisotropic solutions can be considered as a new application of scalar fields in cosmology. The anisotropic field that we will introduce does not really model a quintessence component, because its presence does not accelerate the expansion of the universe. Actually, as we will see, our scalar field produces neither acceleration nor deceleration, so that it can rather be regarded (apart from the anisotropy) as the limit of a quintessence contribution when the acceleration vanishes. Much more importantly, it turns out that the dark energy density associated with this field is similar to the critical density of the model at late times. In this sense, our model suggests that the inclusion of an anisotropic scalar field may provide a mechanism to generate quasiflat [15] universes.

In the case of our anisotropic cosmological solution, the quasiflatness can be rooted in the fact that the energy density of the anisotropic field is proportional to the inverse square of the scale factor, which is precisely the type of dependence that one would expect for the critical density at the final stages of the expansion. The reason is that, in a Bianchi type III universe, there exists a negative contribution of curvature to the energy density, just like in an open FRW model, so that at large cosmological times one would expect the scale factor to be inversely proportional to the Hubble constant and, therefore, to the square root of the critical density.

As we have said, the evolution of the anisotropic solutions that we will present is not accelerated, so that our cosmological model cannot be considered realistic in this respect. One could remedy this situation by including an additional matter source, given by a quintessence field or a cosmological constant. Instead of proceeding in that way, we have preferred to keep the model as simple as possible, both to isolate the cosmological consequences of the anisotropic scalar field and to obtain a solution of Einstein equations in which one can complete all calculations of cosmological parameters using exact expressions.

Apart from clarifying the role that anisotropic scalar fields may play in cosmology, we will also discuss in detail another possible application of our exact anisotropic solution. This application follows from the fact that the spacetime metric of our model is just the vanishing-rotation member of a family of spatially homogeneous, rotating and shear-free metrics that were studied by Korotkiĭ and Obukhov (KO) [8,16]. This family is an expanding version of a class of Gödel-like stationary metrics analyzed by Rebouças and Tiomno (RT) [17]. From now on, we will refer to them as the RTKO metrics. In the limit of small rotation, it is not difficult to employ our exact anisotropic solution to construct approximate solutions for the rotating RTKO cosmologies and supply them with a physically acceptable matter content.

We will also carry out a quantitative analysis of the validity of our approximations when a small rotation is present. In particular, we will find a bound on the angular velocity of the universe guaranteeing that our approximate solutions cannot be distinguished from the unknown exact ones, at least as far as the energy-momentum tensor is concerned.

The RTKO metrics share in fact most of the good properties presented by our exact anisotropic solution with vanishing rotation. For instance, they are all of the Bianchi type III and possess a conformal Killing vector field (CKVF). The existence of a vector of this kind parallel to the four-velocity of dust matter and radiation is known to be the necessary and sufficient condition for the absence of parallax effects [18] and turns out to guarantee the isotropy of the CMB [8]. In addition, the spacetime does not contain closed timelike curves (CTC's) unless the rotation is considerably large [8]. On the other hand, it has been shown that the RTKO metrics reproduce the open FRW metric in the limit of small rotation and nearby distances [19]. All these properties clearly make of the RTKO metrics natural candidates to describe anisotropic cosmological scenarios with rotation.

The rest of the paper is organized as follows. Since the spacetime of the exact cosmological solution that we will construct belongs to the RTKO family, we will first analyze the most relevant properties of these geometries and obtain their Einstein equations in Sec. II. In Sec. III we present our exact anisotropic solution with vanishing rotation, and discuss the corresponding cosmology in Sec. IV. Based on the case of zero angular velocity, we find in Sec. V approximate RTKO solutions with rotation. We also study their cosmological parameters and show that the quasiflatness of the cosmological model persists in the presence of rotation. By comparing the Einstein tensor of our approximate solutions with the energy-momentum tensor of its assumed matter content, we derive in Sec. VI a bound on the angular velocity of the universe ensuring that the relative errors committed in the Einstein equations are small. The conclusions of our work are included in Sec. VII. Finally, an appendix containing some calculations is added.

## II. THE RTKO METRICS

The RTKO metrics are described by the line element

$$ds^2 = a^2(\eta)\left[-(d\eta + le^x dy)^2 + dx^2 + e^{2x}dy^2 + dz^2\right], \quad (2.1)$$

where $\eta$ is the conformal time and $x, y, z$ are the spatial coordinates, all of them assumed to run over the real axis. The parameter $l$, on the other hand, is a constant that can be restricted to be non-negative without loss of generality. From now on, we call it the rotation parameter. The above spacetime does not contain CTC's if and only if $l$ belongs to the interval $[0, 1)$, since it is only then that the metric induced on the sections of constant time is positive definite [8,17]. In the following, we will restrict our considerations to this causal sector, $0 \leq l < 1$.

We will employ the following notation. Greek letters will denote spacetime indices, and the indices $\{0, 1, 2, 3\}$ will designate, respectively, the coordinates $\{\eta, x, y, z\}$. In addition, we adopt units such that $8\pi G = c = 1$, $G$ being Newton constant.

Metric (2.1) is a spatially homogeneous, Bianchi type III metric, with three Killing vector fields given by

$$\xi_{(1)} = \partial_x - y\partial_y, \qquad \xi_{(2)} = \partial_y, \qquad \xi_{(3)} = \partial_z. \quad (2.2)$$

The metric possesses also a CKVF, namely, $\xi^\mu_C = \delta^\mu_0$. Korotkiĭ and Obukhov have proved that, assuming comoving radiation and a comoving observer, the existence of a CKVF guarantees the isotropy of the detected CMB, with the radiation temperature falling with the inverse of the scale factor, and ensures that the redshift of the light coming from astrophysical objects does not depend (explicitly) on the spatial positions of the source and the receiver, but only on the emission and observation times [8,16]. Besides, the CKVF prevents the appearance of parallax effects [18]. Furthermore, in these spacetimes the shear tensor vanishes for comoving observers (with four-velocity equal to $u^\mu = \delta^\mu_0/a$), whereas their rotation tensor $\omega_{\mu\nu}$ is different from zero [7,8]. In particular, their angular velocity is $\omega = \sqrt{\omega_{\mu\nu}\omega^{\mu\nu}/2} = l/(2a)$.

With the help of the coordinate transformation

$$e^x = \cosh r + \cos\phi \sinh r, \tag{2.3}$$
$$ye^x = \sin\phi \sinh r, \tag{2.4}$$
$$\eta = \tilde{\eta} - l\phi + 2l \arctan\left(e^{-r}\tan\frac{\phi}{2}\right), \tag{2.5}$$

one can write metric (2.1) in the cylindrical form [17]

$$ds^2 = -a^2(\eta)\left[d\tilde{\eta} + 2l\sinh^2\left(\frac{r}{2}\right)d\phi\right]^2 + a^2(\eta)\left[dr^2 + \sinh^2 r \, d\phi^2 + dz^2\right]. \tag{2.6}$$

Here, $\tilde{\eta}$ is the new real time, $r$ is non-negative, and $\phi$ is an angular coordinate. Note that the scale factor is not constant on the sections of constant time $\tilde{\eta}$ unless the rotation parameter vanishes [or $a(\eta)$ is a constant number], because $a$ depends on the radial and angular coordinates $r$ and $\phi$ when $l \neq 0$. On the other hand, using the change of coordinates (2.3) and (2.4), one can easily check that the sections of constant time $\eta$ are the direct product of a real line and a two-dimensional pseudosphere.

Let us finally consider the associated Einstein equations. For the diagonal components of the Einstein and energy-momentum tensors, one obtains

$$T^0_0 a^4 = \left(1 - \frac{3l^2}{4}\right)a^2 - 3(1-l^2)\dot{a}^2, \tag{2.7}$$
$$T^1_1 a^4 = T^2_2 a^4 = \frac{l^2}{4}a^2 + (1-l^2)\dot{a}^2 - 2(1-l^2)a\ddot{a}, \tag{2.8}$$
$$T^3_3 a^2 = T^1_1 a^2 + 1 - \frac{l^2}{2}, \tag{2.9}$$

where the overdot denotes the derivative with respect to the conformal time $\eta$. For the non-diagonal components, on the other hand, the result is

$$\begin{aligned} T^1_0 a^4 &= l^2 a\dot{a}, \\ T^2_0 a^4 &= 2le^{-x}(a\ddot{a} - 2\dot{a}^2), \\ T^2_1 a^4 &= -le^{-x}a\dot{a}, \\ T^0_2 a^4 &= (1-l^2)le^x a^2, \\ T^1_2 a^4 &= -(1-l^2)le^x a\dot{a}. \end{aligned} \tag{2.10}$$

The remaining components of the energy-momentum tensor must be identically zero. As far as we know, no physically admissible matter source has been proposed up to date leading to a solution of the above Einstein equations when the scale factor is not constant. Hence, no explicit RTKO cosmological model has been constructed so far. In the next section, we will present an exact solution for the case of vanishing rotation that has an acceptable energy-momentum tensor. This solution represents an anisotropic universe in continuous expansion.

## III. THE ANISOTROPIC SOLUTION

We will now restrict our attention to the RTKO metric obtained when the rotation parameter $l$ vanishes. In the absence of rotation, the Einstein equations require the energy-momentum tensor to be diagonal. The diagonal components must satisfy

$$\epsilon a^4 = 3\dot{a}^2 - a^2, \tag{3.1}$$
$$p_1 a^4 = p_2 a^4 = \dot{a}^2 - 2a\ddot{a}, \tag{3.2}$$
$$p_3 a^2 = p_1 a^2 + 1. \tag{3.3}$$

Here, we have adopted the notation $\epsilon \equiv -T^0_0$ for the energy density and $p_i \equiv T^i_i$ ($i = 1, 2$, or $3$) for the principal pressures of the system.

Let us start by assuming that $\epsilon a^2$ vanishes in the limit of infinite scale factor, as would happen if the matter content consisted exclusively of dust and radiation [6,20]. From Eq. (3.1) one then easily sees that $\dot{a}^2/a^2$ must tend to $1/3$ when $a \to \infty$. Note also that this equation ensures that $a$ increases unboundedly with the conformal time, provided that $\dot{a}$ is initially positive. In addition, supposing that $\epsilon a^2$ is a smooth function of the scale factor, the condition $\epsilon a^2 \to 0$ implies that $(d\epsilon/da)a^3$ vanishes when $a$ becomes infinitely large. Using these facts and taking the time derivative of Eq. (3.1), it follows that the quotient $\ddot{a}/a$ must also tend to $1/3$ when $a$ approaches infinity. Employing now Eq. (3.2), one concludes that $p_1 a^2$ has a finite limit when $a \to \infty$. As a result, the dominant energy condition [21] is violated during the evolution, because for sufficiently large scale factors the pressure $p_1$ becomes larger than the energy density.

Therefore, if we want to reach an acceptable solution of the Einstein equations that respects the energy conditions, we must include matter sources whose energy density does not fall faster than $1/a^2$ when the scale factor expands to infinity. Probably, the simplest way to do this is by introducing an anisotropic massless scalar field minimally coupled to gravity. As we will see below, the corresponding energy density satisfies precisely the minimal requirement of being proportional to the inverse square of the scale factor. Furthermore, the inclusion of such a scalar field will actually suffice to explain all the anisotropies of the model, allowing the rest of the matter content to be isotropic.

In curved spacetime, a massless minimally coupled scalar field satisfies the equation

$$\Phi_{;\mu\nu}g^{\mu\nu} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\Phi_{,\mu}g^{\mu\nu}\right)_{,\nu} = 0, \tag{3.4}$$

whereas its energy-momentum tensor has the form

$$T_{\mu}^{\nu}=\Phi_{,\mu}\Phi_{,\sigma}g^{\sigma\nu}-\frac{1}{2}\Phi_{,\sigma}\Phi_{,\rho}g^{\sigma\rho}\delta_{\mu}^{\nu}. \qquad (3.5)$$

Here, $g$ and $g^{\mu\nu}$ are the determinant and the inverse of the four-metric, the semicolon denotes covariant derivative, and $\delta_{\mu}^{\nu}$ is the Kronecker delta.

Let us then consider an anisotropic scalar field given by $\Phi = Cz$, with $C$ being a constant. This kind of source for the RTKO metrics was already suggested by Rebouças and Tiomno in a stationary context with rotation [17]. It is easily checked that Eq. (3.4) is in fact satisfied by our field in any of the spacetimes (2.1). Besides, from Eq. (3.5), the solution $\Phi = Cz$ has a diagonal energy-momentum tensor, with the following energy density and principal pressures:

$$\epsilon^{(s)}=p_3^{(s)}=-p_2^{(s)}=-p_1^{(s)}=\frac{C^2}{2a^2}. \qquad (3.6)$$

Here, the superindex $(s)$ refers to the contribution of the scalar field.

Note that the corresponding energy density falls with $a^2$, as we had anticipated. In this respect, it is interesting to note that such a kind of decay for the energy density is also expected on the basis of quantum cosmology arguments and might even provide a way to solve the cosmological constant problem [22].

In addition, the principal pressures are now anisotropic. Using this property, it is actually very simple to remove any trace of anisotropy from the Einstein equations of our spacetime. Defining $\epsilon \equiv \bar{\epsilon}+\epsilon^{(s)}$ and $p_i \equiv \bar{p}_i + p_i^{(s)}$, we see from Eq. (3.3) that the anisotropic contributions of the model can be absorbed in the scalar field just by imposing that $C^2 = 1$. Since the orientation of $z$ can be inverted at will (producing an apparent flip of sign in the constant $C$), we will fix from now on $\Phi = z$.

Equations (3.1)-(3.3) become then

$$\bar{\epsilon}a^4 = 3\dot{a}^2-\frac{3}{2}a^2, \qquad (3.7)$$

$$\bar{p}a^4 = \dot{a}^2-2a\ddot{a}+\frac{a^2}{2}, \qquad (3.8)$$

where $\bar{p} = \bar{p}_i$ for any $i = 1, 2$ or 3. Remarkably, these are exactly the Einstein equations of an open FRW model with curvature parameter $\kappa$ equal to $-1/2$ [6]. Equivalently, they can be written as the equations of the standard FRW model with $\kappa = -1$ under the scaling:

$$\eta \equiv \sqrt{2}\eta_F$$
$$\sqrt{2}\, a\Big(\eta=\sqrt{2}\eta_F\Big) \equiv a_F(\eta_F). \qquad (3.9)$$

Here, the subindex $F$ denotes the conformal time and scale factor of the open FRW cosmology. Notice that these relations imply that the cosmological time of our model coincides with that of the standard FRW spacetime, because $a d\eta = a_F d\eta_F$.

From the above comments, it should be clear that the evolution of the scale factor in our model reproduces the expansion found in an open FRW cosmology, except for some qualitatively irrelevant scalings by factors of the order of the unity. Owing to this fact, and leaving aside the anisotropy of the sections of constant time, the cosmological solution that we will construct leads essentially to the same history of the universe as a standard open FRW scenario, at least during the epoch in which the scalar field has a negligible contribution to the energy and pressure of the system. Furthermore, regarding the anisotropy of the spatial sections, we recall that our spacetime metric can be written in the form (2.6) with $l = 0$. In fact, since such metric reduces to an open FRW metric in the limit of nearby distances $r \ll 1$ [19], no differences should be expected in physical processes or observations which do not involve distant regions.

We are now in an adequate position to obtain the solution of the Einstein equations that we were seeking. In addition to the anisotropic scalar field, we suppose that the matter content is given by radiation and dust, as it is usually done in standard FRW cosmology. We will describe these matter sources by a two-component perfect fluid, with comoving four-velocity $u^{\mu} = \delta_0^{\mu}/a$. The assumption that the radiation present in the system adopts the form of a comoving perfect fluid, together with the properties of the RTKO metrics [8], guarantees that the CMB of the model is isotropic. Similarly, the fact that the dust matter can be treated as a comoving perfect fluid ensures the applicability of Hubble law (in the leading-order approximation) to any kind of radiation that could be emitted by dust particles, since the radiation frequency varies then just like the inverse of the scale factor [8]. For such a matter content, the expression of the energy and pressure that appear in Eqs. (3.7) and (3.8) are

$$\bar{\epsilon}=\frac{A^2}{a^4}+\frac{D}{a^3}, \qquad \bar{p}=\frac{A^2}{3a^4}, \qquad (3.10)$$

where $A$ and $D$ are two non-negative constants. The first term on the right-hand side of these equations corresponds to the radiation component, whereas the dust matter contributes only to the energy density [6,20].

With the above energy and pressure, Eq. (3.7) turns out to be a first integral of Eq. (3.8), and admits a unique increasing solution that vanishes at $\eta = 0$. The exact solution is given explicitly by

$$a=\frac{D}{3}\left[\cosh\left(\frac{\eta}{\sqrt{2}}\right)-1\right]+\sqrt{\frac{2}{3}}A\sinh\left(\frac{\eta}{\sqrt{2}}\right). \qquad (3.11)$$

This expression can be inverted in $\eta \geq 0$, obtaining

$$\eta=\sqrt{2}\ln\left[\frac{3a+D+\sqrt{9a^2+6Da+6A^2}}{D+\sqrt{6}A}\right]. \qquad (3.12)$$

On the other hand, integrating $dt = a d\eta$, we arrive at the following expression for the cosmological time:

$$t = \frac{D}{3}\left[-\eta + \sqrt{2}\sinh\left(\frac{\eta}{\sqrt{2}}\right)\right] + \frac{2A}{\sqrt{3}}\left[\cosh\left(\frac{\eta}{\sqrt{2}}\right) - 1\right]. \tag{3.13}$$

From the last two formulas, one can also calculate $t$ as a function of the scale factor.

## IV. THE ANISOTROPIC COSMOLOGICAL MODEL

In the preceding section, we have constructed an exact solution of Einstein equations that describes an expanding universe containing an anisotropic massless scalar field and a comoving perfect fluid composed of radiation and dust matter. The spacetime metric is given by the element of the family (2.1) with vanishing rotation. As a consequence of the properties of the RTKO metrics, the CMB of the model is isotropic and the redshift of the radiation emitted by the comoving dust depends only on the emission and observation times [8]. We have also seen that our anisotropic metric coincides with the metric of an open FRW universe in the limit of nearby distances. Moreover, the conformal expansion of our solution reproduces (apart from some trivial scalings) the evolution encountered in a standard open FRW cosmology with matter content formed exclusively by isotropic dust and radiation. Hence, the history of the universe in our anisotropic model parallels that of an open FRW solution, at least as far as the scalar field does not supply the dominant contribution to the energy-momentum tensor.

Like in the analogue FRW cosmology with energy density and pressure given by $\bar{\epsilon}$ and $\bar{p}$, the radiation dominated era of our anisotropic model corresponds to the epoch with $0 \leq a \leq A^2/D$. At small times, the universe expands from an initial singularity following exactly the same evolution law as in standard FRW cosmology [6,20], namely,

$$a = \frac{A\eta}{\sqrt{3}} = \sqrt{\frac{2At}{\sqrt{3}}}. \tag{4.1}$$

This behavior can be easily obtained from Eq. (3.11) in the region $\eta \ll 1$. As a particular consequence, the Hubble parameter and the energy density adopt, at the initial stages of the expansion, the expressions $H \equiv \dot{a}/a^2 = 1/(2t)$ and $\bar{\epsilon} = 3/(4t^2)$, which coincide with the result of the standard model in the radiation era. In particular, it follows that the initial relative energy density is $\Omega = \bar{\epsilon}/(3H^2) = 1$.

When $a$ increases beyond $A^2/D$, the dust component starts to supply the major contribution to the energy density and the universe enters a dust dominated era with an evolution of the scale factor similar to that presented in an open FRW cosmology. Such era ends when the energy of the anisotropic scalar field becomes the most important matter component. This occurs when $\epsilon^{(s)}$ equals the dust energy density, i.e., when $a = 2D$. We assume that $A \ll \sqrt{2D}$, so that there exists a sufficiently large epoch $A^2/D \leq a \leq 2D$ dominated by matter with strong clustering properties. As far as $a < 2D$, the contribution of the anisotropic scalar field is subdominant, and the model leads essentially to the same cosmological predictions as an open FRW model.

For scale factors larger than $2D$, the anisotropic scalar field dominates the evolution. The expansion is then of the approximate form

$$a = \left(\frac{A}{\sqrt{6}} + \frac{D}{6}\right)\exp\left(\frac{\eta}{\sqrt{2}}\right) = \frac{t}{\sqrt{2}}, \tag{4.2}$$

as one can check from Eq. (3.11) by analyzing the sector of large times. Note that this evolution is linear in the cosmological time, like at the final stages of an open FRW model. This was in fact expected, because the time dependence of the scale factor must always be similar to that of an open FRW universe without scalar field, as we showed in the preceding section. In the limit $\eta \to \infty$, the Hubble parameter displays then the behavior $H = 1/t$, and the energy density is $\epsilon^{(s)} = H^2$. Hence, at large times, the relative energy density becomes $\Omega = 1/3$.

Actually, from Eq. (3.11) we can derive the exact expressions of the Hubble parameter, the deceleration parameter $q$, and the relative energy density [6] at all times of the evolution. We get

$$H \equiv \frac{\dot{a}}{a^2} = \sqrt{\frac{3a^2 + 2Da + 2A^2}{6a^4}}, \tag{4.3}$$

$$\Omega \equiv \frac{\epsilon}{3H^2} = \frac{a^2 + 2Da + 2A^2}{3a^2 + 2Da + 2A^2}, \tag{4.4}$$

$$q \equiv 1 - \frac{a\ddot{a}}{\dot{a}^2} = \frac{Da + 2A^2}{3a^2 + 2Da + 2A^2} > 0, \tag{4.5}$$

where we have used that $\epsilon = \epsilon^{(s)} + \bar{\epsilon}$ and employed Eqs. (3.6) and (3.10). We recall that the parameter $q$ is positive when the expansion decelerates.

In the limits $a \to 0$ and $a \to \infty$ (i.e., when $\eta$ tends to zero and infinity, respectively), we recover from these equations the behavior discussed above for $H$ and $\Omega$. Furthermore, it is not difficult to prove that Eq. (4.4) defines a strictly decreasing function of the scale factor, $\Omega(a)$. Since the universe is always expanding in our solution, we conclude that the relative energy density of our model suffers a continuous decrease from its initial unit value at the big-bang singularity, reaching the asymptotic lower bound of one-third in the limit of large times. In this way, the contribution of the anisotropic scalar field guarantees that the energy density of the model is of the order of the critical one during the whole evolution, leading to a quasiflat universe.

Obviously, the model is not fully realistic; in particular, the positivity of Eq. (4.5) means that the expansion decelerates in our solution, contradicting the present observations of SNe Ia [12]. The result $q > 0$ can be easily understood on the basis of our matter content: as we have seen, the scalar field leads to a uniform expansion, linear in the cosmological time, whereas the presence of radiation and dust decelerates the expansion. Note, however, that the deceleration is similar to that found in a standard open FRW cosmology without cosmological constant and quintessence fields. This follows from the fact that the deceleration parameter $q$ reflects only the time dependence of the scale factor, and this dependence coincides in our solution and in an open FRW model.

In order to attain an accelerated expansion in our anisotropic scenario, we could simply add a positive cosmological constant $\Lambda$ to the matter content. Indeed, it is easy to check that, for the epoch in which $\Lambda$ dominates the energy density, Eq. (3.1) would lead to an exponential expansion. Like in standard FRW cosmology, however, we have preferred to analyze here the case without cosmological constant (or quintessence) because in this way we can obtain an explicit solution that allows us to perform all calculations to conclusion. In addition, the inclusion of other matter sources would have prevented us from clearly isolating the consequences of the anisotropic scalar field.

In order to estimate the values of the parameters $A$ and $D$ and the present values of $a$, $t$, $q$, and $\Omega$ in our model, we can proceed as follows. From Eq. (4.3) we get

$$a_0 = \sqrt{\frac{3}{6H_0^2 - 2\bar{\epsilon}_r - 2\bar{\epsilon}_d}}, \tag{4.6}$$

where the subindex 0 means evaluation at the present time, and the subindices $r$ and $d$ denote the radiation and dust components of the matter content. In addition, if $\Omega_d$ is the contribution of dust matter to the relative energy density and $Z_{eq}$ is the redshift corresponding to the equilibrium between dust and radiation, we have that $\bar{\epsilon}_d = 3\Omega_d H_0^2$ and $\bar{\epsilon}_r = \bar{\epsilon}_d(1 + Z_{eq})^{-1}$. Finally, $A^2 = \bar{\epsilon}_r a_0^4$ and $D = \bar{\epsilon}_d a_0^3$. With these values and formulas (4.4) and (4.5), we can also determine the quantities $q_0$ and $\Omega_0$. Using (approximately) the values of the concordance model [23] for the present Hubble parameter and relative energy density of pressureless matter, $H_0 = 65$ km/(sMp) and $\Omega_d = 0.35$, as well as $Z_{eq} + 1 = 5000$, we obtain that $A = 1.6 \times 10^{24}$ m, $D = 1.0 \times 10^{26}$ m, $a_0 = 1.2 \times 10^{26}$ m, $t_0 = 12$ Gyr, $q_0 = 0.18$, and $\Omega_0 = 0.57$.

From these estimates, we see that the assumption $A \ll \sqrt{2}D$ is actually satisfied in our solution. The dust era corresponds to the interval of scale factors $2.5 \times 10^{22}$ m $\leq a \leq 2.0 \times 10^{26}$ m, which is large enough for structure formation and contains the present period of the evolution. We also see that the equilibrium between dust matter and the anisotropic scalar field would be reached when $a = 2D = 2.0 \times 10^{26}$ m, a value of the scale factor that is only slightly larger than the present one. Thus, in our cosmological model, we would be almost at the end of the dust dominated epoch.

It is worth noting that, although the CMB of the model is isotropic and the redshift of the radiation emitted by dust particles depends only on the value of the scale factor at the moment of emission, and not on the spatial position of the source, the fact that the metric is anisotropic implies that the distance to astrophysical objects with identical redshift varies with the direction of observation. One might then worry about the compatibility of this anisotropy with the available data about extra-galactic sources at high redshift, e.g. with the apparent isotropy detected in the Hubble diagram for SNe Ia at redshifts of order unity. In order to discuss this issue, let us consider the angular diameter distance [6], which can be defined by the relation $dA_e = r_a^2\, d\Omega_0$ [24]. Here, $dA_e$ is the (infinitesimal) intrinsic perpendicular area of the source, which subtends the solid angle $d\Omega_0$ at the origin where, using the homogeneity of the spacetime, we locate the receiver [24,25]. The luminosity distance is then $r_l = r_a(1 + Z)^2$, with $Z$ being the redshift of the source [25]. Hence, one only has to care about the anisotropies that appear in $r_a$. Using the expressions given in Ref. [24] (or just applying the formulas of Ref. [26]), it is possible to show that

$$r_a^2 = a^2(\eta_e)\ (\eta_0 - \eta_e)^2\ Y[\sin\theta(\eta_0 - \eta_e)], \tag{4.7}$$

where $Y(u) \equiv \sinh u/u$, $\theta \in [0, \pi]$ is the angle formed by the line of sight and the $z$ axis, and $\eta_e$ is the conformal time of emission. As anticipated, $r_a$ depends on the direction of observation and, for fixed $Z$ (and present time $\eta_0$), its maximum $r_M$ and minimum $r_m$ are reached when $\sin\theta$ equals the unity or tends to zero, respectively. The magnitude of the relative variation of $r_a$ on the celestial sphere can be described with the quantity $\varepsilon_a = (r_M - r_m)/r_m$. Employing Eqs. (3.12), (4.7), and $1 + Z = a_0/a(\eta_e)$, it is straightforward to see that $\varepsilon_a$ increases with $Z$. More importantly, substituting the values of the constants $A$, $D$, and $a_0$ obtained above, one can check that the relative variation of the angular diameter distance is only of the order of 5% for $Z = 1$, while for $Z = 2$ $\varepsilon_a$ is close to 10%. These variations do not seem to conflict with the observational data, and do not dominate over the systematic and statistical uncertainties, evolution effects, and experimental errors that are present in the determination of astronomical distances.

Finally, let us point out that the age of the universe in our model ($t_0 = 12$ Gyr), although very close, is still beyond the lower bounds obtained from radioactive dating of stars [27] or studies of globular clusters [28]. These results show that (except for the absence of acceleration and the corresponding quintessence contribution to the relative energy density) our anisotropic model is at least compatible with the main features of modern standard cosmology.

## V. APPROXIMATE ROTATING SOLUTIONS

In this section, we will present a generalization of the solution (3.11) for a non-vanishing rotation parameter, $l \neq 0$. We will assume the same matter content as in the absence of rotation, namely, a two-component perfect fluid, formed by radiation and dust, and an anisotropic scalar field $\Phi = z$ minimally coupled to gravity. For small values of the parameter $l$, we will see that the RTKO metric that we will obtain can be regarded as an approximate solution of the Einstein equations. In this way, one can construct an approximate cosmological model describing the expansion of a rotating anisotropic universe which contains isotropic background radiation. Actually, supposing that $l$ is sufficiently small, the inclusion of rotation produces only small corrections in the cosmological model constructed in Sec. IV. As a consequence, our approximate solutions will lead to a similar cosmology, both qualitatively (apart from the existence of an angular velocity) and quantitatively.

The energy-momentum tensor will have the form

$$T_{\mu}^{\nu} = (\bar{p} + \bar{\epsilon})u^{\nu}u_{\mu} + \bar{p}\delta_{\mu}^{\nu} + (T^{(s)})_{\mu}^{\nu}, \tag{5.1}$$

where $u^{\mu}$ is the four-velocity of the two-component fluid, its energy density and pressure are given in Eq. (3.10), and $T^{(s)}$ denotes the energy-momentum tensor of the anisotropic scalar field. The components of this diagonal tensor appear in Eq. (3.6) (with $C = 1$). The parameters $A$ and $D$, which determine the energy density, are assumed to be exactly the same as in the solution with vanishing rotation. Like in that case, we also consider comoving perfect fluids with $u^{\mu} = \delta_0^{\mu}/a$.

Using the general RTKO non-diagonal metric (2.1), we obtain the covariant four-velocity $u_{\mu} = -a(\delta_{\mu}^{0} + le^{x}\delta_{\mu}^{2})$. Then, from our definition (5.1), we see that the diagonal components of the energy-momentum tensor are (formally) the same as in our solution with zero angular velocity, whereas all the non-diagonal components vanish except $T_2^0$. This last component takes the expression

$$T_2^0 = -le^{x}\left(\frac{4A^2}{3a^4} + \frac{D}{a^3}\right). \tag{5.2}$$

Let us first consider the diagonal time component (2.7) of the Einstein equations. When the rotation parameter does not vanish, this equation has the following solution for our value of the energy density:

$$a = \frac{D}{3X_l}\left[\cosh\left(\sqrt{\frac{X_l}{2Y_l}}\,\eta\right) - 1\right] + \sqrt{\frac{2}{3X_l}}A\sinh\left(\sqrt{\frac{X_l}{2Y_l}}\,\eta\right), \tag{5.3}$$

where we have introduced the definitions

$$X_l \equiv 1 - \frac{l^2}{2}, \qquad Y_l \equiv 1 - l^2. \tag{5.4}$$

The above scale factor increases with the conformal time in $\eta \geq 0$ and vanishes at $\eta = 0$. In addition, it reproduces Eq. (3.11) when $l$ vanishes. Note also that, since we have imposed that $l \in [0, 1)$, the ranges of $X_l$ and $Y_l$ are, respectively, $(1/2, 1]$ and $(0, 1]$.

Substituting the above time dependence of the scale factor and the expression of the energy-momentum tensor in Eqs. (2.8) and (2.9), it is easy to check that the Einstein equation $G_3^3 = T_3^3$ is satisfied exactly; however, the other diagonal spatial components of the Einstein tensor differ by a term $l^2/(2a^2)$ from their assumed values. In other words,

$$G_1^1 - T_1^1 = G_2^2 - T_2^2 = \frac{l^2}{2a^2}. \tag{5.5}$$

Concerning the non-diagonal components (2.10) of the Einstein equations, it is not difficult to prove using Eq. (5.2) that, when $l$ is small, the Einstein tensor of the analyzed RTKO metric provides an approximate solution up to terms of the order of $l^2$ for $G_0^1 = T_0^1$ and of order $l$ for the rest of equations. Therefore, we conclude that the difference between the components of the energy-momentum tensor of our system and those of the Einstein tensor of the metric (2.1) and (5.3) vanish at least as fast as $l$ when $l \to 0$, and become, in general, negligible when the rotation parameter is small. In Sec. VI we will use this fact to set an upper bound to the global angular velocity in order to ensure that the relative error committed in the energy-momentum tensor with our approximation is smaller than a certain quantity.

Let us now analyze the behavior of our approximate cosmological solutions with rotation. Inverting relation (5.3), we obtain the conformal time

$$\eta = \sqrt{\frac{2Y_l}{X_l}}\ln\left[\frac{3X_l a + D + \sqrt{9X_l^2a^2 + 6X_lDa + 6X_lA^2}}{D + \sqrt{6X_l}\,A}\right], \tag{5.6}$$

and, integrating $dt = a d\eta$, we get the following expression for the cosmological time:

$$t = \frac{D}{3X_l}\left[-\eta + \sqrt{\frac{2Y_l}{X_l}}\sinh\left(\sqrt{\frac{X_l}{2Y_l}}\,\eta\right)\right] + \frac{2A}{X_l}\sqrt{\frac{Y_l}{3}}\left[\cosh\left(\sqrt{\frac{X_l}{2Y_l}}\,\eta\right) - 1\right]. \tag{5.7}$$

From relation (5.3), one can also derive the Hubble parameter, the deceleration parameter, and the relative energy density of our approximate solutions:

$$H = \sqrt{\frac{3X_la^2 + 2Da + 2A^2}{6Y_la^4}}, \tag{5.8}$$

$$\Omega = \frac{Y_l(a^2 + 2Da + 2A^2)}{3X_la^2 + 2Da + 2A^2}, \tag{5.9}$$

$$q = \frac{Da + 2A^2}{3X_la^2 + 2Da + 2A^2} > 0. \tag{5.10}$$

These formulas replace Eqs. (4.3), (4.4), and (4.5), respectively, when the rotation differs from zero.

In the limit $a \to 0$, we get again $H = 1/(2t)$ and $q \to 1$, as in the standard cosmological model. In this limit, the relative energy density takes the value $\Omega = Y_l = 1 - l^2$, so that $\epsilon = 3(1 - l^2)/4t^2$. The expansion and history of the primordial universe is therefore affected only by corrections of the order of $l^2$ [see also Eq. (5.3)]. On the other hand, in the sector of large scale factors $a \to \infty$, one can easily check that $H = 1/t$ and $q \to 0$, just like on the exact solution presented in Sec. III. At this final stage of the expansion, the relative energy density tends to $Y_l/(3X_l)$, a limit which is positive for $l \in [0, 1)$ and differs from the value of 1/3, corresponding to the non-rotating case, by terms of the order of $l^2$, supposing that the rotation parameter is small.

Finally, it is not difficult to prove that the relative energy density (5.9) is a strictly decreasing function of the scale factor. Like in the model discussed in Sec. IV, $\Omega$ remains then bounded away from zero during the whole evolution, the lower bound being its positive limit when $a \to \infty$. Actually, if $l \ll 1$, the energy density is always of the same order of magnitude as the critical one. Therefore, we see that the introduction of an anisotropic scalar field leads to a quasiflat universe even in the presence of rotation.

## VI. VALIDITY OF THE APPROXIMATION

In this section we want to carry out a quantitative analysis of the error committed in Einstein equations by identifying the energy-momentum tensor (5.1) with the Einstein tensor of the RTKO metric whose scale factor is the time function (5.3). More specifically, we want to show that it is possible to set an upper bound to the rotation parameter (and hence to the present angular velocity) so that the relative error in our estimation of the energy-momentum tensor is smaller than a fixed quantity.

For each component of the Einstein equations, we define the relative error introduced with our approximation as the quotient $|G_\mu^\nu - T_\mu^\nu|/\epsilon$, where $\epsilon \equiv -T_0^0$ is the energy density of the matter content. We want to analyze under which circumstances these relative errors are smaller than a given number $\Delta$. Since, for any reasonable approximation, all relatives errors should be at least smaller than the unity, we assume that $\Delta < 1$ from now on. As we have seen, the only non-trivial components of the Einstein equations that are exactly solved by the evolution law (5.3) are those corresponding to $G_0^0$ and $G_3^3$. For the remaining components, the error is at most of the order of $l$ when the rotation parameter is small.

Concerning our definition of relative errors, it is clear that $\epsilon$ is the largest diagonal component of the energy-momentum tensor. In addition, we will see below that, in the spacetime region and range of parameters of physical interest, the other non-vanishing component of this tensor (namely, $T_2^0$) is also smaller than the energy density. Therefore, with our definition, we are just comparing the errors made in the estimation of the energy-momentum tensor with its dominant component.

To analyze these errors, we need to deal with factors of the form $e^{\pm x}$ that appear in most of the non-diagonal components of the Einstein tensor, as can be seen in Eqs. (2.10). In doing this, we will proceed as follows. Since the model is spatially homogeneous, we can always locate the observer at the origin. From a physical point of view, the only phenomena that can affect the observer at a generic, present time $\eta_0$ are those that occurred in the spacetime region that is causally connected with him. Thus, from now on we will restrict our discussion to that region. Let us also suppose that we are only interested in events that happened in a certain interval of time $\eta \in [\eta_1, \eta_0]$, with $0 \le \eta_1 < \eta_0$. Although we will make $\eta_1 = 0$ at the end of our calculations, we prefer to leave this number free for the moment in order to allow for other possibilities.

In a RTKO spacetime, one can check that the maximum absolute value that the coordinate $x$ can take at time $\eta < \eta_0$ in the region that is causally connected with the origin at present is $(\eta_0 - \eta)/\sqrt{1 - l^2}$. A point at time $\eta$ with this value of $x$ is connected with the origin at $\eta_0$ by the null geodesic with vanishing $z$ and $dy/d\eta = le^{-x}/(1 - l^2)$. Hence, the region of the spacetime that we want to analyze is contained in

$$\left\{ x \in I_\eta \equiv \left[ \frac{-\eta_0 + \eta}{\sqrt{1 - l^2}}, \frac{\eta_0 - \eta}{\sqrt{1 - l^2}} \right], \quad \eta \in [\eta_1, \eta_0] \right\}. \quad (6.1)$$

In particular, for each fixed value of $\eta$, the extrema of the interval $I_\eta$ correspond to points that are causally connected with the observer.

Moreover, the above region is invariant under the reversal $x \to -x$. Using this fact and recalling that $l \in [0, 1)$ and $D \ge 0$, it is possible to show that, among all the conditions coming from the requirement that the relative errors be smaller than the quantity $\Delta$, the most restrictive condition is that corresponding to the non-diagonal component $T_0^2$ of the energy-momentum tensor. This component leads to the inequality

$$\frac{le^{-x}}{Y_l} \frac{6X_l a^2 + 6Da + 8A^2}{3a^2 + 6Da + 6A^2} \le \Delta, \quad (6.2)$$

where $a = a(\eta)$ is given by Eq. (5.3), the pair of coordinates $(\eta, x)$ must belong to the region (6.1), and we have adopted again the notation (5.4).

On the other hand, from expression (5.2), we get

$$\frac{|T_2^0|}{\epsilon} = le^x \frac{6Da + 8A^2}{3a^2 + 6Da + 6A^2}. \quad (6.3)$$

Recalling that the region under analysis is invariant under a flip of sign in the coordinate $x$ (and that $X_l > 0$, $Y_l \le 1$ and $\Delta < 1$), we then see that condition (6.2) ensures that, in the region of physical interest, the energy density dominates over the non-diagonal component $T_2^0$

of the energy-momentum tensor, as we had commented above.

In addition, note that, since $e^{-x}$ is a strictly decreasing function of $x$, its maximum value for $x \in I_\eta$ is obtained at $(-\eta_0 + \eta)/\sqrt{1-l^2}$. So, the most stringent condition contained in Eq. (6.2) is

$$\frac{l}{Y_l} \leq \exp\left[\frac{\eta(a) - \eta(a_0)}{\sqrt{Y_l}}\right] \frac{3a^2 + 6Da + 6A^2}{6X_l a^2 + 6Da + 8A^2}\Delta. \quad (6.4)$$

We have employed here relation (5.6) to write the conformal time in terms of $a \in [a_1, a_0]$, with $a_0 > a_1$. These two values of the scale factor are reached, respectively, at the present time $\eta_0$ and at the initial time of our considerations $\eta_1$.

Using the explicit form of the function $\eta(a)$, it is actually possible to show that, for fixed parameters $l$ and $\Delta$, the right-hand side of the above inequality is an increasing function of $a$. As a consequence, its minimum value in the interval $[a_1, a_0]$ is attained when $a = a_1$. In this way, we conclude that the necessary and sufficient condition for the relative errors to be smaller than $\Delta$ in the region of physical relevance is obtained from Eq. (6.4) by making $a = a_1$. In particular, if we consider the whole region that can be causally connected with the origin since the initial big bang, i.e. $a_1 = 0$, we get

$$\frac{l}{Y_l} \leq \exp\left[\frac{-\eta(a_0)}{\sqrt{Y_l}}\right] \frac{3}{4}\Delta, \quad (6.5)$$

where we have employed that $\eta$ vanishes when $a = 0$.

This inequality sets an upper bound to $l$, beyond which our solution cannot be considered a good approximation modulo relative errors smaller than $\Delta$. It is worth noticing that the conformal time $\eta(a_0)$ that appears in Eq. (6.5) depends on the rotation parameter $l$, as well as on the constants $A$ and $D$, via relation (5.6). Owing to this dependence, it is in general difficult to find the exact value of the upper bound on $l$ once the scale factor $a_0$ and the numbers $\Delta$, $A$, and $D$ are known. In the Appendix, we present a method to estimate such an upper bound with great accuracy. In practice, nevertheless, it is possible to get a really good estimate by simply replacing $Y_l = 1 - l^2$ with the unity and substituting $\eta(a_0)$ by the value $\eta_0$ of the present conformal time corresponding to the exact solution with vanishing rotation parameter. It is not difficult to check that these approximations amount to disregarding corrections of the order of $l^2$ in the upper bound on $l$. Employing the values of $a_0$, $A$, and $D$ given in Sec. IV, one arrives in this way at

$$l \leq 0.0337\Delta. \quad (6.6)$$

As we have said, a more careful procedure to estimate this upper bound is presented in the Appendix, where we also consider the possibility $a_1 = a_0/1500$, corresponding approximately to the time of decoupling between dust and radiation, and a model with slightly different cosmological parameters, $\Omega_d = 0.3$ and $H = 70$ km/(sMpc). In all these cases, we obtain a value of the upper bound which is close to the result given above.

From inequality (6.6), we can easily derive an upper bound on the global angular velocity at present. Using that $\omega = l/(2a)$ and $a_0 = 1.2 \times 10^{26}$ m (the value obtained in Sec. IV), we get $\omega \leq 4.1 \times 10^{-20}\Delta$ s$^{-1}$. Thus, in order to have a relative error $\Delta \leq 2.5\%$ one needs to impose, approximately, that $\omega \leq 10^{-21}$ s$^{-1}$, while a more permissive error $\Delta \leq 25\%$ would lead to $\omega \leq 10^{-20}$ s$^{-1}$.

Up to date, there exists no well-established and generally accepted estimate of the angular velocity of the universe. In models with shear, some upper bounds can be inferred from the CMB anisotropy, but these bounds do not apply to the shear-free RTKO spacetimes. There are some estimations of $\omega$ based on the observed rotation of the plane of polarization of cosmic electromagnetic radiation [8,16,29], leading to $\omega \sim 10^{-18}$ s$^{-1}$. However, such observations are very controversial, and the derived value of $\omega$ could well be two or three orders of magnitude smaller [24].

An independent estimate $\omega \sim 10^{-21}$ s$^{-1}$ can be obtained from the analysis of the rotation of galaxies [30]. This result agrees with another estimation that is not based on observation, but on a heuristic argument, namely, the extension to the problem of rotation of the large number hypothesis put forward by Dirac. The angular momentum of the observed universe is $L \sim \rho\omega a^5$, where $\rho$ is the density of matter. From the large number hypothesis, we get $L \sim \hbar\Lambda_D^3$ [31], where $\hbar$ is Planck constant and $\Lambda_D \sim 10^{39}$ is Dirac scaling parameter [32]. So, we have $\omega \sim \hbar\Lambda_D^3/(\rho a^5)$. With $\rho = 3 \times 10^{-27}$ kg/m$^3$ and $a = a_0 = 1.2 \times 10^{26}$ m, this leads to $\omega \sim 10^{-21}$ s$^{-1}$.

Let us finally remark that the upper bound that we have obtained for $l$ is only aimed at determining the interval of rotation parameters in which the approximate RTKO solution presented in Sec. V is acceptably good. In principle, rotating solutions with larger angular velocities are possible, but their energy-momentum tensor cannot be approximated by the matter content considered here. On the other hand, additional restrictions on the rotation parameter $l$ could come from the requirement that the anisotropies that arise in the formulas of the luminosity and angular diameter distances are compatible with the observational data. The consideration of these anisotropies, however, cannot be carried out analytically if $l \neq 0$, because, by contrast with the situation found in the case with vanishing rotation (see Sec. IV), the exact dependence of these distances with the redshift $Z$ is not manageable anymore. What is available now is (the first terms of) their Kristian-Sachs expansion in powers of $Z$ [25]. Using the expressions given by Obukhov for this expansion [24] and defining the relative variation of the angular diameter distance $\varepsilon_a$ like in Sec. IV, it is possible to show that $\varepsilon_a \simeq 2l$ up to second order corrections in $Z$ and in the rotation parameter. Therefore, recalling the bound on $l$ obtained above, we can affirm that the influence of rotation in the formulas for distances is negligible, at least as far as we do not consider sources of high redshift. For high redshifts the Kristian-Sachs expansion is expected not to be valid, and a more careful analysis is needed to determine the relevance of the anisotropies.

## VII. CONCLUSIONS

In this paper, we have shown that it is possible to construct anisotropic models that are at least compatible with the main features of standard cosmology. In particular, we have found an exact solution of Einstein equations which describes an expanding universe containing an anisotropic scalar field and a comoving perfect fluid with two components: radiation and dust. The solution is spatially homogeneous, but the sections of constant time are anisotropic, its topology being the product of a pseudosphere and a real line. Even so, the background radiation is perfectly isotropic and the redshift experimented by any possible emission of the dust particles varies with the scale factor like in a FRW model. Moreover, the expansion is conformal and follows the same evolution law as in a standard open FRW spacetime filled with dust and radiation.

The relation between the redshift of astronomical sources and their angular diameter (or luminosity) distance turns out to be anisotropic, because so is the spacetime metric. However, this anisotropy does not conflict with the current observational data, because the corresponding variation of distances with the line of sight in our model is not dominant compared with the systematic and experimental errors of the measurements.

The introduction of the massless, anisotropic scalar field leaves, nevertheless, one important imprint: the energy density of the model is of the order of the critical density at all times. Therefore, the universe is always quasiflat. In more detail, the relative energy density equals the unity at the initial big-bang singularity, like in FRW cosmology, and decreases monotonically during the whole evolution to a lower bound of one third, which is the asymptotic limit reached at infinitely large times.

The cosmological model that we have constructed is not completely realistic because, for instance, it does not predict the observed accelerated expansion of the universe. In principle, this defect could be cured by including additional dark energy in the system, supplied either by a cosmological constant or by a quintessence field. This modification of our model will be discussed elsewhere. Here, we have concentrated our attention in our simple model because it permits a clear discussion of the effects of the anisotropic scalar field and allows to obtain explicitly the time dependence of the scale factor and the cosmological parameters.

We have also presented a quite straightforward application of our exact solution, namely, the obtention of approximate cosmological models describing spatially homogeneous, anisotropic spacetimes with rotation. This has been possible because the anisotropic metric of our exact solution is in fact the element with vanishing rotation of a family of shear-free rotating metrics with remarkable properties, including the isotropy of the comoving CMB and the preservation of the standard relation between the redshift of light and the value of the scale factor when this light was emitted.

Assuming that the matter content is the same as in our exact non-rotating solution, we have proved that it is possible to generalize the time dependence of the scale factor so as to attain an approximate solution of Einstein equations in the presence of rotation. More specifically, if one restricts all considerations to the causal past of the observer, we have shown that the error committed with our approximations in Einstein equations, relative to the energy density of the system (which is the dominant component of the energy-momentum tensor), remains smaller than any required quantity $\Delta$ if one sets an upper bound linear in $\Delta$ to the angular velocity of the present universe. In particular, we have calculated this bound using the values of the Hubble parameter and the relative energy density of pressureless matter provided by the concordance model [23]. For relative errors of a few percent, the upper bound that we have found turns out to be of the same order of magnitude as those obtained from observation of the rotation of galaxies [30] and heuristic considerations involving the large number hypothesis [31].

Finally, an interesting possibility would be to analyze the angular power spectrum of primordial fluctuations in the CMB of these anisotropic cosmologies. This analysis would require an extension of the standard scheme of cosmological perturbation theory [2] that dealt with the fact that the spatial sections of the spacetime are not maximally symmetric, took into account the anisotropic dependence of distances on the redshift, and treated the rotation parameter also in a perturbative manner. These issues will be the subject of future research.

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## APPENDIX

In this appendix, we will estimate the upper bound that inequality (6.4), evaluated at $a = a_1$, sets to the rotation parameter $l$. Remembering expression (5.6), we can write the considered inequality as

$$\frac{l}{(1-l^2)} \leq J(X_l, X_l)\Delta, \tag{A1}$$

where

$$J(U,V) \equiv \frac{3a_1^2+6Da_1+6A^2}{6Ua_1+6Da_1+8A^2} \times \left( \frac{3Ua_1+D+\sqrt{9U^2a_1^2+6UDa_1+6UA^2}}{3Ua_0+D+\sqrt{9U^2a_0^2+6UDa_0+6UA^2}} \right)^{\sqrt{2/V}}. \quad \text{(A2)}$$

Note that $J$ depends on the non-negative constants $A$ and $D$ and on the values of the scale factor at present, $a_0$, and at the initial time, $a_1$.

It is straightforward to see that, for $a_0 > a_1$, $J(U,V)$ increases with $V$, assuming that $U$ and $V$ are positive. Since, according to Eq. (5.4), $X_l$ ranges in $(1/2, 1]$, it then follows that a necessary condition for Eq. (A1) to be satisfied is that $l/(1-l^2) \leq J(X_l, 1)\Delta$. In addition, one can check that $l/(1-l^2)$ is greater than $J(X_l, 1)$ when $l$ approaches the unity, whereas the opposite happens at $l = 0$, provided that $A > 0$. Therefore, the functions $l/(1-l^2)$ and $J(X_l, 1)$ intersect each other at least once in $l \in [0, 1)$. Moreover, in this interval of $l$, both functions turn out to be strictly increasing. It is then possible to prove that the largest of the intersection points,

$$L \equiv \max \left\{ l \in [0,1): \quad \frac{l}{1-l^2} = J(X_l, 1) \right\}, \quad \text{(A3)}$$

can be obtained by numerical iteration. Namely, defining $l_1 = 1$ and $l_{n+1} = f(l_n)$, one can get $L$ as the limit of the sequence $\{l_n\}$, where $f(l) \equiv F[J(X_l, 1)]$ and

$$F[J] \equiv \frac{\sqrt{1+4J^2}-1}{2J}. \quad \text{(A4)}$$

Recalling then that $\Delta < 1$ and $1 - l^2 \leq 1$, one easily concludes that a necessary condition for inequality (A1) to hold is

$$l \leq J(X_L, 1)\Delta. \quad \text{(A5)}$$

Let us now find a sufficient condition ensuring inequality (A1). From our previous discussion, we already know that $l \leq L$ and that $J(U,V)$ increases with $V$ if $U$ and $V$ are positive. Employing the definition of $X_l$, we then see that $J(X_l, X_l) \geq J(X_l, X_L)$. In addition, $J(X_l, X_L)$ is an increasing function of $l$ in the interval $[0, 1)$, regardless of the constant value of $X_L \in (1/2, 1]$. So, $J(X_l, X_L) \geq J(1, X_L)$, since $X_l$ becomes the unity at $l = 0$. Hence, it follows that a sufficient condition for Eq. (A1) to hold is $l/(1-l^2) \leq J(1, X_L)\Delta$ or, equivalently $l \leq F[J(1, X_L)\Delta]$. Finally, taking into account that $0 < J(1, X_L)\Delta < 1$ for all the allowed values of $\Delta$ and $X_L$, and that $F[J] \geq J(1-J^2)$ if $0 \leq J \leq 1$, it is easy to derive the simpler sufficient condition

$$l \leq J(1, X_L)[1 - J^2(1, X_L)]\Delta. \quad \text{(A6)}$$

Using the values of $a_0$, $A$ and $D$ obtained in Sec. IV, making $a_1 = 0$ (i.e., considering the entire causal past of the origin), and following the procedure explained above to determine the value of $L$, one can check that the necessary and sufficient conditions given in Eqs. (A5) and (A6) lead in fact to coincident upper bounds on $l$, up to the third significant figure. With this degree of accuracy, one gets the bound $l \leq 0.0337\Delta$, which reproduces in fact the estimate reached in Sec. VI. If one made instead $a_1 = a_0/1500$, paying thus attention only to those events in the causal region which occurred (approximately) after the time of decoupling, one would obtain, with the same level of precision, $l \leq 0.0442\Delta$.

In order to check the sensibility of our estimates to the particular values adopted for the relative energy density of dust matter and the Hubble parameter, we have repeated the evaluation of the constants $A$ and $D$, the present scale factor $a_0$, and the upper bound on $l$ taking $\Omega_d = 0.3$ and $H_0 = 70$ km/(sMpc). In this case, following the arguments explained in Sec. IV, one gets $A = 1.3 \times 10^{24}$ m, $D = 7.2 \times 10^{25}$ m, and $a_0 = 1.1 \times 10^{26}$ m, which are close to the values found with $H_0 = 65$ km/(sMpc) and $\Omega_d = 0.35$. In addition, with $a_1 = 0$, Eqs. (A5) and (A6) lead now to the bound $l \leq 0.0260\Delta$ (again up to the third significant figure), whereas $l \leq 0.0344\Delta$ if $a_1 = a_0/1500$. So, the upper bound reached for $l$ is of the same order of magnitude in all the considered cases.

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